The Operator-Theoretic Renormalization Group – QED-lite

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Credits & Contents

Report on work with:

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and others.

1. Spectral Theory – Theory of Spectra
2. The “Standard Model” of Electrons, (Nuclei) and Photons (Pauli-Fierz Model)
3. Atomic Bound States and Resonances (BFS), etc.
4. Analytical Methods: RG based on the Feshbach-Schur Map
5. Properties of (Effective) Time Evolution
6. Conclusions

1. Spectral Theory ⟷ Theory of Spectra

- Etymology of "spectral theory"; (Balmer – Ritz, Rydberg – Bohr – Heisenberg et al. – ...)
- H : atomic (or molec.) Hamiltonian; no coupling to e.m. field

Eigenvalues: $e_0 < e_1 < e_2 < \cdots$

How does one gather info. on $\{e_i\}_{i=0,1,2,\cdots}$?

Don't neglect coupling of atom (molec.) to e.m. field;

c.c. $\alpha \approx 1/137$ small $\rightarrow$

problem "perturbative"?

Eigenstates corresp. to e_i, $i \geq 1$, turned into meta-stable states; ($\nearrow$ FGR-cond.)

- Bohr's F.C.:

$$\hbar\omega_{nm} \approx e_n - e_m$$

But: Lamb-shift $O(\alpha)$

How to understand this?

- Fermi's G.R.:

Life times, $O(\alpha)$, $< \infty$.

- Isolated atoms prepared in "bound state" relax to groundstate by emission of photons ⟷ Rayleigh scattering

  How does space-time history of relaxation look like? (Eff. dynamics)

- Ionization of atoms by light; …; Compton scatt. of electrons; …

What's the math. to understand spectroscopy?

→ Modern spectral theory, "RG improved perturbation theory" (BFS, Pizzo, BFP, ...)

Fundamental problems:

- "Naïve" p.th. divergent at finite order: infra-red divergence!
- Apparently no analyticity in α at $\alpha = 0$.
- Moreover, eff. dynamics of atoms dissipative.
- Ionization non-perturbative

2. "Standard Model" of Electrons (Nuclei) & Photons

(P-F)

For simplicity:

- Static nucleus
- Only one electron
- Electron spin (magn. moment) neglected

Because we can't do better.

UV cutoff imposed on interactions between electrons & photons.

Thus:

$$\mathcal{H} = L^2(\mathbb{R}^3, dx) \otimes \mathcal{F},$$

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \left(L^2(\mathbb{R}^3, dk) \otimes \mathbb{C}^2\right)^{\otimes_s n}$$

$$H = \left(-i\nabla + \alpha^{3/2} A_\Lambda(\alpha x)\right)^2 + V(x) + H_f$$

$$A_\Lambda(x) = (2\pi)^{-3/2} \sum_{\lambda=\pm} \int dk \, \frac{\chi_\Lambda(k)}{\sqrt{2|k|}} \times$$

$$\left[\varepsilon_\lambda(k) a_\lambda(k) e^{ik\cdot x} + h.c.\right]$$

$a_\lambda(k), a_\lambda^*(k)$: annihilation- & creation operators on $\mathcal{F}$.

$$H_f = \sum_{\lambda=\pm} \int dk \, a_\lambda^*(k) |k| a_\lambda(k)$$

H is self-adjoint on $\mathcal{D}(H) \subsetneq \mathcal{H}$; $\mathcal{D}(H) = \mathcal{D}(H_{\alpha=0})$ if $e^2 = \alpha$ small enough.

Dilatation analyticity

a: generator of dil. on $\mathcal{H}$.

$$\left(a = \left[-\frac{i}{2}(x\cdot\nabla + \nabla\cdot x)\right] - d\Gamma\left(\frac{i}{2}k\cdot\nabla_k + \frac{i}{2}\nabla_k\cdot k\right)\right)$$

$$U(\theta) := e^{i\theta a} \quad ;$$

$$\mathcal{D}_{\theta_0} := \{\psi\in\mathcal{H} \mid \psi(\theta) := U(\theta)\psi \text{ analytic for } |\theta|<\theta_0\} \subsetneq \mathcal{H}.$$

<u>Hyp</u>. $\chi_\Lambda(k)$ smooth approx. of $1(|k|\le\Lambda)$, analyt. in k, $|\mathrm{Im}\,k|<e^{\theta_0}$

[V dil. analytic]

$\Rightarrow$ $H(\theta) := U(\theta)HU(-\theta)$ analyt family of ops.

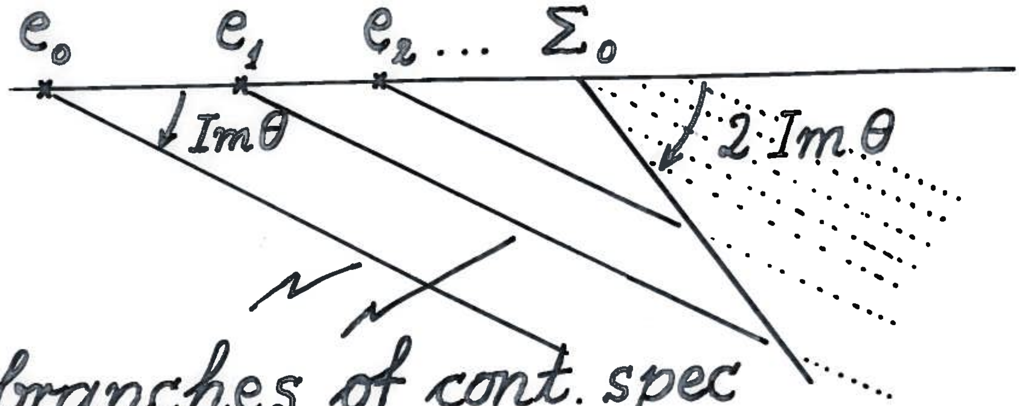
Then

$$\left\langle \psi, \frac{1}{z-H}\psi \right\rangle \overset{\theta \text{ real}}{=} \left\langle \psi(\theta), \frac{1}{z-H(\theta)}\psi(\theta) \right\rangle$$

$$\overset{\psi \in \mathcal{D}_{\theta_0}}{=} \left\langle \psi(\bar{\theta}), \frac{1}{z-H(\theta)}\psi(\theta) \right\rangle,$$

if $|\mathrm{Im}\,\theta| < \theta_0$.

spec $H_{\alpha=0}(\theta)$:



For $H_{\alpha=0}(\theta)$, $\mathrm{Im}\,\theta \neq 0$, diff. branches of cont. spec separated; e_i at tip of branch.

infit. version: Mourre theory.

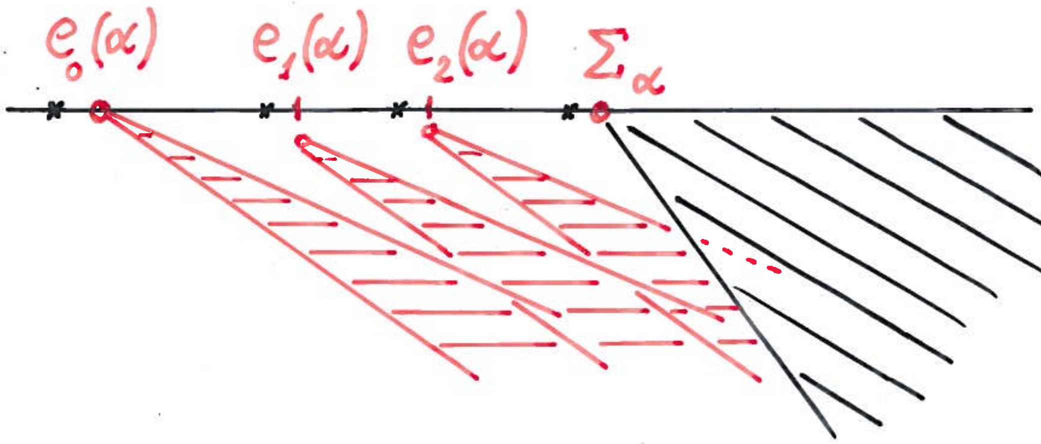
3. Bound States and Resonances, etc. (BFS)

What happens if $\alpha \neq 0$?

$$\Sigma_\alpha = \Sigma_0 + O(\alpha^3)$$

If $\psi \in \operatorname{Ran} 1(H \leq \Sigma_\alpha - \delta)$ then ψ exp. well loc. near nucl., decay rate $\approx \sqrt{\delta}$.

$\operatorname{spec} H_{\alpha \neq 0}(\theta)$, $\operatorname{Im} \theta > 0$:



$e_0(\alpha)$: non-deg. eigenvalue

(BFS, GLL ← F)

- $\mathrm{Re}\, e_i(\alpha) - e_i$: Lamb shift (Be
  $\mathrm{Im}\, e_i(\alpha)$: Fermi GR,
  $i \geq 1$.

- $e_i(\alpha)$ indep. of θ; are eigenvalues of $H(\theta)$, $\mathrm{Im}\,\theta > 0$, α small enough
  (BFS)

  ψ_j ~ unpert. eigenstate corresp. to e_j.

- Survival probability:

$$|\langle \psi_j, e^{-itH} \psi_j \rangle| = e^{-|\mathrm{Im}\, e_j(\alpha)|t} + O(\alpha^{\cdots}) \text{ power law.}$$

- Nature of spectrum:

$$J_\alpha := (e_0(\alpha), \Sigma_\alpha - \delta),\ \delta > 0.$$

$$\sigma(H) \cap J_\alpha = \sigma_{ac}(H) \cap J_\alpha .$$

This + limited abs. princ. can also be derived using Mourre th. + RG; (FGS).

- If V, or ext. magn. field time-dep.: Adiabatic thm. for ground st. (AE, T) and resonances (A-SF).

On states $\psi \in \mathrm{Ran}\, 1(H \leq \Sigma_\alpha - \delta)$,

- $\exists$ asymptotic em fields:

$$\prod_{\ell} F^{\pm}_{\mu\nu}(h^{\mu\nu}_{\ell})\psi$$

$$= \operatorname{s-lim}_{t\to\pm\infty} e^{itH} \prod_{\ell} F_{\mu\nu}(h^{\mu\nu}_{\ell,-t}) e^{-itH}\psi$$

(FGS, G)

→ Møller ops. for Rayleigh scattering; (asy. completeness: FGS, FaS, DGK)

- Perturbation th. is asy. to scattering ampl. to arb. order (BFP)

↓

Bohr's frequency cond.

4. Analytical Methods, I: Feshbach-Schur Map

$H\psi = z\psi$, P: orth. proj.

Then

$$(PHP + PHP^{\perp})\psi = zP\psi \qquad i$$

$$(P^{\perp}HP + P^{\perp}HP^{\perp})\psi = zP^{\perp}\psi \qquad ii$$

Let $\overline{H} := P^{\perp}HP^{\perp}$, $z \in \rho(\overline{H})$.

Then *ii* $\Rightarrow$

$$P^{\perp}\psi = (z - \overline{H})^{-1}P^{\perp}HP\psi$$

& *i* $\Rightarrow$

$$F_P(H - z)\psi = 0, \qquad iii$$

$$F_P(H-z) := P(H-z)P - \qquad iv$$
$$- PHP^{\perp}(\overline{H}-z)^{-1}P^{\perp}HP$$

Properties of F-S Map:

(1) $z \in \sigma_{pp}(H) \Leftrightarrow 0 \in \sigma_{pp}(\mathcal{F}_P(H-z))$

(2) $z \in \rho(H) \Leftrightarrow 0 \in \rho(\mathcal{F}_P(H-z))$

(3) $H\psi = z\psi \Leftrightarrow \mathcal{F}_P(H-z)\varphi = 0,$

$$\psi = Q(z)\varphi \qquad \varphi = P\psi$$

$$Q(z) = P + (z-\overline{H})^{-1}P^{\perp}HP$$

(4) $P(H-z)^{-1}P = \mathcal{F}_P(H-z)^{-1}$

(5) If $P' < P$

$$\mathcal{F}_{P'}(H-z) = \mathcal{F}_{P'}(\mathcal{F}_P(H-z))$$

& Isospectrality

Application:

$$H = H(\theta),\ \mathrm{Im}\,\theta > 0,\ P_0 = P^{at}_{e_i} \otimes 1(H_f \leq 1)$$

$$i \geq 1$$

$$z \in D_{\delta_0}(e_i), \quad \delta_0 = 1 - O(\alpha^{\cdots})$$

$$H^{(0)} := \mathcal{F}_{P_0}(H(\theta) - z) \qquad (6)$$

acting on $1(H_f \leq 1)\mathcal{F}$

Apply resolvent exp. & re-Wick ordering (+ "pull-through formula") to R.S. of (6):

$$H^{(0)} = T^{(0)}(z) + \sum_{\substack{n,m \\ n+m \geq 1}} W^{(0)}_{n,m}(z) + \mathcal{E}^{(0)}(z),$$

$$T^{(0)}(z) = T(H_f, z), \quad \left| T(\varepsilon, z)/\varepsilon \right| \leq 1 + c_0,$$

$$W_{n,m}(z) = 1(H_f \leq 1) \int d\underline{k} \int d\underline{\tilde{k}} \prod_1^n a^*(k_i) \times$$

$$\times \underbrace{w_{n,m}(\underline{k}, \underline{\tilde{k}}; z)} \prod_1^m a(\tilde{k}_j)\, 1(\cdots)$$

bd. by $\prod_1^n |k_i|^{-1/2+\mu} \prod_1^m |\tilde{k}_j|^{-1/2+\mu}$

$\mathcal{E}^{(0)}(z)$: scalar, analyt. fu. of z,

with $\mathcal{E}^{(0)}(z^{(0)}) = 0$,

for some $z^{(0)} \in D_{\delta_0}(e_i)$, $\operatorname{Im} z^{(0)} < 0$.

Next, apply F–S map, with

$P = P_1 = 1(H_f \leq \rho)$, $0 < \rho < 1$, to $H^{(0)}$,

with $z \in D_{\delta_1}(z^{(0)}) \subset D_{\delta_0}(e_i)$:

$$H^{(1)} := \mathcal{F}_{P_1}(H^{(0)})$$

$$= T^{(1)}(z) + \sum_{\substack{n,m \\ n+m \geq 1}} W^{(1)}_{n,m}(z) + \mathcal{E}^{(1)}(z)$$

$$\| W^{(1)}_{n,m}(z) 1(H_f \leq \rho) \| \leq cst. \, \xi^{n+m} \rho^{1+\mu},$$

$0 < \xi < 1$, $\ldots$, $\mathcal{E}^{(1)}(z^{(1)}) = 0$,

for some $z^{(1)} \in D_{\delta_1}(z^{(0)})$, $\operatorname{Im} z^{(1)} < 0$.

. . . .

Inductive construction:

$P_k = 1\left(H_f \leq \rho^{(1+\frac{\mu}{2})^{k-1}}\right)$,

ρ small enough (dep. on μ),

α " " (" " μ).

$\rightarrow H^{(k)}, D_{\delta_k}(z^{(k-1)}) \subset \cdots \subset D_{\delta_1}(z^{(0)}) \subset D_{\delta_0}(e_1)$

with $H^{(k)} \simeq (1+c_k) H_f \, 1(H_f \leq \rho^{(1+\frac{\mu}{2})^{k-1}})$,

$c_k \rightarrow c_\infty$, $|c_\infty| < 1$,

$\delta_k \rightarrow 0$ superexpon.

$z^{(k)} \rightarrow e_i(\alpha)$, $\operatorname{Im} e_i(\alpha) < 0$,

for $i \geq 1$; ($\operatorname{Im} e_0(\alpha) = 0 \Rightarrow$

$e_0(\alpha)$ eigenvalue of H_α!)

All this clearly works for

complex c.c., too (→ certain analyticity props.; albeit not in $\alpha^{1/2}$, for standard model).

"Smooth F-S map": Replace $1(H_f \leq (\cdot))$ by C^∞-approx. to char. fu.. Complicates alg., but simplifies analysis. Can be combined with Mourre theory (FGS).

Obsession: Cast all this in form of RG map on Banach space of Hamiltonians: l.o.

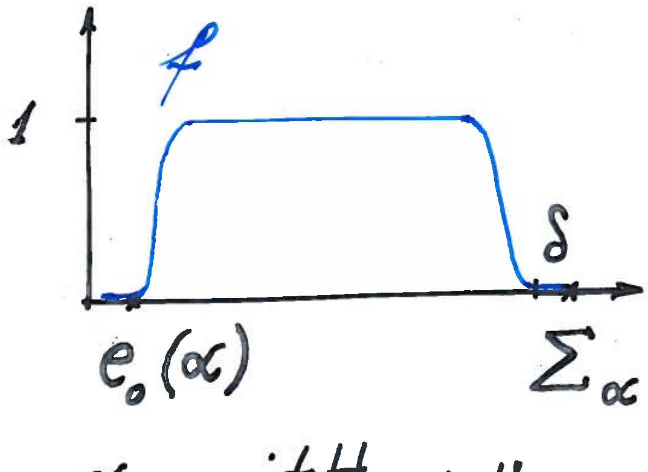
5. Properties of Time Evol.

$>\alpha$ small $<$

1. Survival prob. & life times of resonances: As above (Hunziker's arguments).

2. Escape of photons: Mourre th. $\Rightarrow$ limited absorption principle; namely, for

$$\psi = f(H_\alpha)\varphi\,,\quad \varphi \in D(a^\alpha)$$



$$\| \langle a \rangle^{-\alpha} e^{-itH_\alpha} \psi \| \underset{t\to\infty}{\longrightarrow} 0$$

3. Relaxation to groundstate

Asy. completeness of Rayleigh scattering (FGS, Fau-Si, DeR-Kup)

$$\overset{FGS}{\Longrightarrow} \langle e^{-itH_\alpha}\psi, a e^{-itH_\alpha}\psi\rangle$$

$$\underset{t\to\infty}{\longrightarrow} \langle \psi_\alpha^{(0)}, a\,\psi_\alpha^{(0)}\rangle,$$

$\forall a \in \mathcal{A}_{loc.}$, provided

$$\psi \in \mathrm{Ran}\, P_{H_\alpha}([e_0(\alpha), \Sigma_\alpha - \delta])$$

and

$$e^{-itH_\alpha}\psi \underset{t\to\infty}{\simeq} \sum_i \prod_{l_i} F^+_{\mu\nu}(h^{\mu\nu}_{l_i,t})\,\psi_\alpha^{(0)}$$

4. Rel. to g.s. & R to E in gen. spin-boson models

$H = H_{at} + H_f + \lambda D$ on

$\mathcal{H} = \mathbb{C}^N \otimes \mathcal{F}$;

$H_{at} = M \otimes \mathbb{1}$,

M an $N \times N$ matrix w. e.v.'s

$$e_0 < e_1 < \cdots < e_{N-1}$$

D : FGR cond. $\Rightarrow$ all e_i, $i \geq 1$, move off real axis in 2nd order p.t.; (see above). Then

for small enough λ

$$\langle e^{-itH}\psi, a\, e^{-itH}\psi \rangle \xrightarrow[t\to\infty]{} \langle \psi^{(0)}, a\, \psi^{(0)} \rangle,$$

$\forall a \in \mathcal{A}_{loc}$!

$\langle e^{-itH}\psi, N_f\, e^{-itH}\psi \rangle$

unif. bounded in t;

(FGS $\longrightarrow$ De Roeck-Kup.)

+ Return to Equilibrium

(Jaksić-Pillet, BFS, ...)

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5. Quantum Brownian motion (De Roeck-F)

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6. Conclusions

"In all my films, I have been faithful to these suspension points in the conclusions. Besides, I have never written the word 'END' on the screen."
(Federico Fellini)

"Everyone wants to understand art (physics)*. Why don't we try to understand the song of a bird? Why do we love the night, the flowers, everything around us, without trying to understand them? But in the case of a painting* (result in physics)*, people think they have to understand." (Pablo Picasso)*

Thank you for listening!

My Manifesto

I propose that, at all decent institutions of higher education, *one or two days per semester* will be declared to be

Days of Reflection and of Protest!

During these days, we will not teach or attend committee meetings, and there won't be any exercise classes. Instead, we will discuss some of the serious problems threatening our civilization, draft declarations and reach out to the media, with the aim to make it clear to all circles wielding power that we no longer accept – (to mention some examples among others):

My Manifesto, ctd.

- That internal tensions and conflicts in countries, such as the *Ukraine*, are “solved” by armed conflicts rather than by political dialogue and compromise;
- that innocent people are slaughtered in ugly civil wars and by terrorist activities, such as those in Syria and Iraq;
- that countries threaten other countries with warfare;
- that weapons are sold to (clans) in countries plagued by civil war or other forms of unrest and conflict;
- that religions are abused for purposes of power and suppression of people;
- that the dignity and the rights of women are abused and offended in the name of religion;

My Manifesto, ctd.

- that people are harassed or killed because of their race or faith;
- that nothing is done against the perversions of 21st Century Capitalism;
- that the resources of Planet Earth continue to be looted shamelessly.

These are *but some examples of numerous problems* threatening the survival of humankind in peace and dignity. –

Where is the *"Peace Movement"*, where are movements such as *"Occupy Wall Street"*, *"Survivre et Vivre"*? What is the *"Club of Rome"* doing? Why are the media silent about the activities of these and other groups? Why do academics not have a strong voice in political debates, anymore?

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Students and Academics raise your voices, arise!